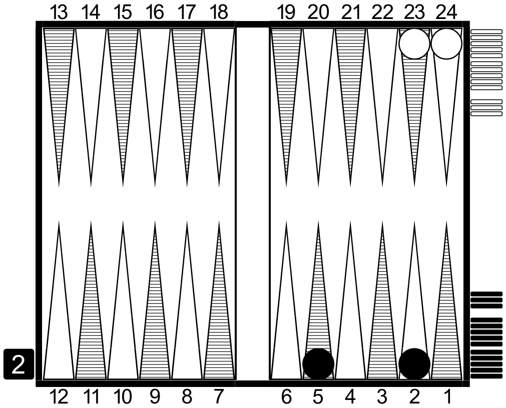
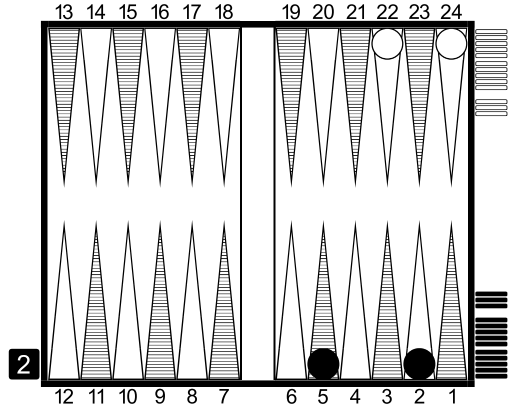
## SOLUTIONS – WEEK #12 (3/2/21)[1]

**POSITION #1**



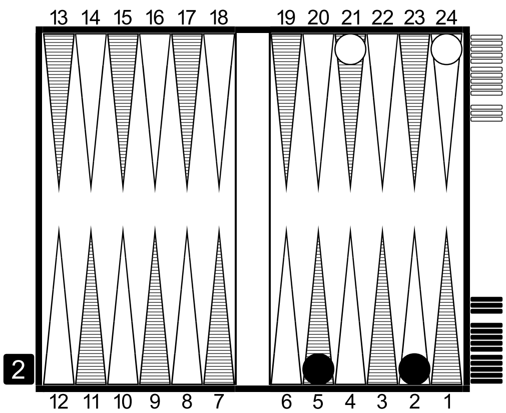

**REDOUBLE/TAKE**

**POSITION #2**



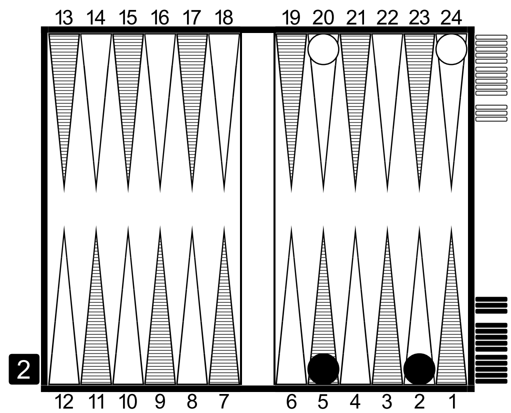

**BORDERLINE REDOUBLE/TAKE**

**POSITION #3**



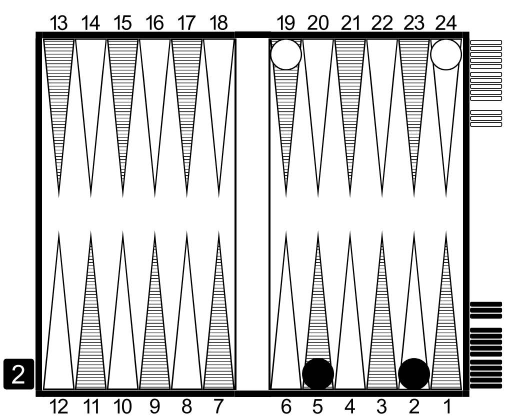

**NO REDOUBLE/TAKE**
**(JACOBY PARADOX)**

**POSITION #4**



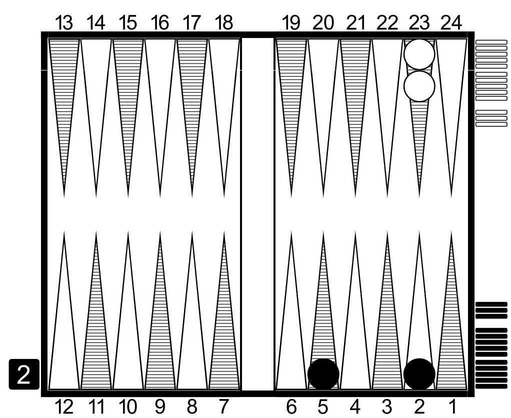

**REDOUBLE/TAKE**

**POSITION #5**




**REDOUBLE/PASS**

**POSITION #6**

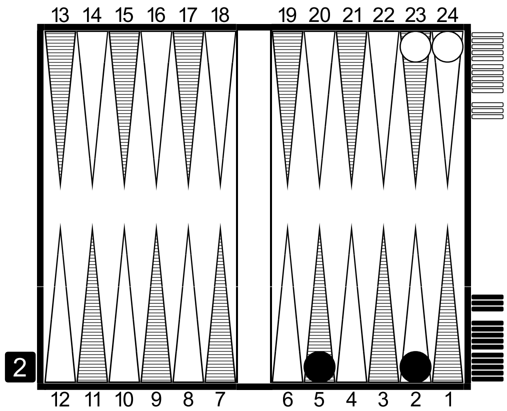



**NO REDOUBLE/TAKE**
**(JACOBY PARADOX)**

[1] Note that throughout this solution write-up, green shading is used in the equity tables to highlight the outcome that results when both players make their best play.

# DISCUSSION/ANALYSIS

**POSITION #1**




**REDOUBLE/TAKE**

Here, Black is a favorite to finish his bear-off on his next roll. Of the 36 possible rolls, 19 take the two checkers off (66, 55, 44, 33, 22, 65, 64, 63, 62, 54, 53 and 52), while 17 miss (11, 61, 51, 43, 42, 41, 32, 31 and 21). Since this is essentially a "last roll" situation and Black is a favorite, Black should turn the cube to double the stakes.

White can take because he wins more than 25% of the time, which is the break-even point for taking a double in a "last roll" situation. To see why this is the break-even point, imagine the stakes are $1 per point and that the players replay the position four times. If White passes the redouble, he loses $2 in each game. On the other hand, if he takes and wins 25% of the time (or 1 game in 4), then he'll win with the four-cube one time, and lose with the four-cube three times. Over four games, he'll win $4 once and lose $4 three times. His net will be a loss of $8 over the four games, or a loss of $2 per game – the same amount lost if he just passes the cube.

**EQUITIES – POSITION #1**

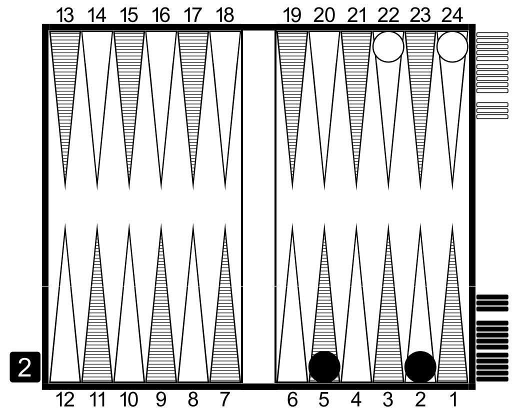
**IF BLACK DOUBLES**

| | |
|---|---|
| Black Winning Pct. | 52.78% |
| White Winning Pct. | 47.22% |
| Black $ per Game | $0.222 |

**IF BLACK HOLDS ONTO THE CUBE**

| | |
|---|---|
| Black Winning Pct. | 52.78% |
| White Winning Pct. | 47.22% |
| Black $ per Game | $0.111 |

**POSITION #2**




**BORDERLINE REDOUBLE/TAKE**

We now move White's checker on the deuce-point to the 3-point. If Black redoubles, he will do just as well as he does in Position #1, since he loses if he misses on this roll (White will redouble and Black will have to pass, because White only misses with a roll of 21, so Black doesn't have anywhere near the 25% winning chances that he needs to take White's recube after he misses).

On the other hand, if Black holds the cube and doesn't double, he has an additional way to win the game if he misses. White could also miss by rolling 21, after which Black gets a second chance to take his checkers off (in fact, he doesn't need to even roll the dice again if White misses, as he can just turn the cube and force White to pass).

So, Black does just as well as he does in Problem #1 when he turns the cube, but he does a bit better than he does in Problem #1 when he doesn't turn the cube.

**EQUITIES – POSITION #2**

**IF BLACK DOUBLES**

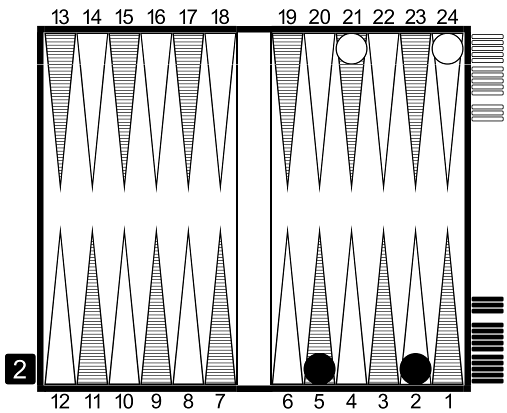
| | |
|---|---|
| Black Winning Pct. | 52.78% |
| White Winning Pct. | 47.22% |
| Black $ per Game | $0.222 |

**IF BLACK HOLDS ONTO THE CUBE**

| | |
|---|---|
| Black Winning Pct. | 55.40% |
| White Winning Pct. | 44.60% |
| Black $ per Game | $0.216 |

As the equity table above shows, although Black wins slightly more often when he doesn't double, he still wins more money by doubling and raising the stakes. The correct cube action is for Black to double and for White to take. The double is very close, though (a difference of less than one cent per game).

**POSITION #3**




**NO REDOUBLE/TAKE (JACOBY PARADOX)**

Now we move the White checker one-pip further back. Interestingly, Black no longer has a redouble. This is the case even though White's position is worse in this position than it is in Position #1, where Black has a clear double![2] The reason for this result is not that Black will do worse by doubling in this position than he does in Position #1 – in fact, he doesn't, he does just as well. When Black redoubles, he creates what is essentially a "last roll" position that he will win on his first roll on 19 of 36 shakes.

But suppose that Black doesn't redouble here. Now, if Black misses on his first throw, White will also miss with 7 of his 36 possible rolls (32, 31, 21 and 11). This is more than 75% of the time – so if Black had given White the cube in this position, White wouldn't have needed to take a roll – he could redouble and Black would have been forced to pass. So, by not redoubling here, in addition to winning if he can bear off his checkers on his first throw, Black will also win the games if he misses and then White rolls one of his 7 missing numbers (again, Black won't even have to take a second roll, he can just turn the cube and force White to pass).

The relevant equities are as follows:

**EQUITIES – POSITION #3**

**IF BLACK DOUBLES**

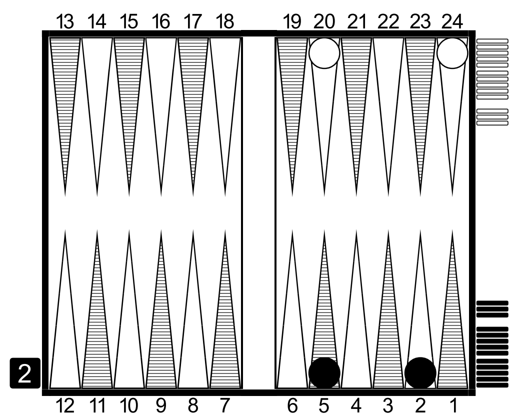
| | |
|---|---|
| Black Winning Pct. | 52.78% |
| White Winning Pct. | 47.22% |
| Black $ per Game | $0.222 |

**IF BLACK HOLDS ONTO THE CUBE**

| | |
|---|---|
| Black Winning Pct. | 61.96% |
| White Winning Pct. | 38.04% |
| Black $ per Game | $0.478 |

Black wins over 60% of the time by holding onto the cube. This gives him more equity than he gets by doubling, since he is only a slight favorite when he doubles. Holding onto the cube is clearly the best play in this position and wins more money for Black.

**POSITION #4**




**REDOUBLE/TAKE**

We now move the White checker back one more pip to the 5-point. Based on the analysis of the previous position, it would seem that Black should again hold onto the cube, since his winning chances from holding the cube will go up still further, with White now having 13 rolls that miss (43, 42, 41, 32, 31, 21 and 11), instead of just 7.

[2] This paradoxical result (where Black does not have a proper redouble, but making White's position *stronger* would result in him having a proper redouble) is known as the **Jacoby Paradox**. It is named for Oswald Jacoby, who presented it in his book, ***The Backgammon Book***. See Jacoby, Oswald, and Crawford, John R., ***The Backgammon Book***, 1970, The Viking Press, pp. 116-117.

Instead, Black once again should redouble. Why does this happen? What's different about this position? The answer is that in this position, Black has better winning chances and he can now take White's redouble to eight! White misses with 13 rolls, so Black has more than the 25% winning chances that he needs to take a recube from White if he fails to clear his board on his first roll.

**EQUITIES – POSITION #4**

**IF BLACK DOUBLES**

| | |
|---|---|
| Black Winning Pct. | 69.83% |
| White Winning Pct. | 30.17% |
| Black $ per Game | $1.062 |

**IF BLACK HOLDS ONTO THE CUBE**

| | |
|---|---|
| Black Winning Pct. | 69.83% |
| White Winning Pct. | 30.17% |
| Black $ per Game | $0.793 |

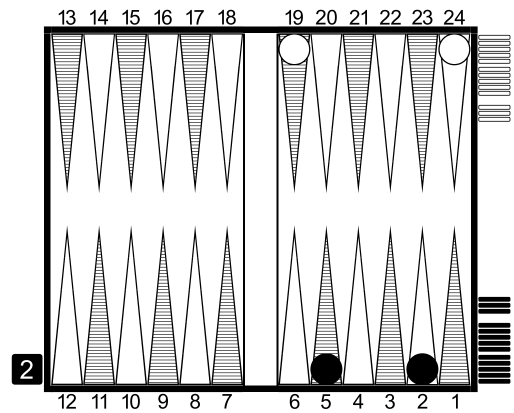
If Black holds onto the cube, he averages a win of more money than he does in the previous position ($0.793 per game, as compared to $0.478 per game). But now that he can take a recube from White, and his game winning chances are up to almost 70%, he can do even better for his wallet in this case by turning the cube himself first, doubling the stakes.

I won't get into the details that underlie the math calculations here, but he'll lose $8 when White wins (about 30% of the time), and the remainder of the time he'll win either $4 or $8 for himself. It works out that he averages $1.062 per game by redoubling, as compared to the lower amount of $0.793 per game that he wins if he holds onto the cube.

Essentially, while maintaining ownership of the cube has some value in its own right, this value is overshadowed in this position by the value that Black gets by raising the stakes in a position where he is a good-sized favorite. Black should redouble.

Of course, White has an easy take – losing an average of $1.062 per game is much better than passing and losing an immediate $2.00. White should also redouble whenever Black misses on his initial roll, since he will then become the favorite in what is essentially a "last roll" position.

**POSITION #5**




**REDOUBLE/PASS**

This position is simpler to analyze than the others. White only has 15 rolls that bear his two checkers off (66, 55, 44, 33, 22, 65, 64, 63, 62 and 61). This is less than half of his possible rolls. White can only win on his next shake if Black misses and then White rolls one of his 15 numbers that bear his two checkers off, or chances of 17/36 * 15/36. This amounts to 19.68%. As this is less than 25%, Black should just redouble and White should pass.

Failing to redouble here would be a pretty big mistake by Black as it allows White a chance to win a game that should immediately end in Black's favor. Likewise, a take on White's part would also be a pretty big mistake as he wins the game less than 20% of the time.

**EQUITIES – POSITION #5**

**IF BLACK DOUBLES AND WHITE PASSES**

| | |
|---|---|
| Black Winning Pct. | 100.00% |
| White Winning Pct. | 0.00% |
| Black $ per Game | $2.000 |

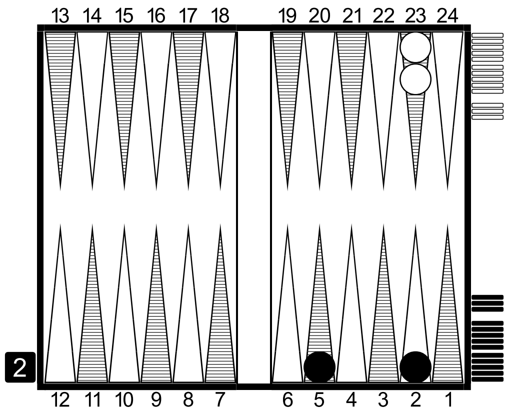
**IF BLACK DOUBLES AND WHITE TAKES**

| | |
|---|---|
| Black Winning Pct. | 80.14% |
| White Winning Pct. | 19.86% |
| Black $ per Game | $2.412 |

**IF BLACK HOLDS ONTO THE CUBE**

| | |
|---|---|
| Black Winning Pct. | 80.32% |
| White Winning Pct. | 19.68% |
| Black $ per Game | $1.213 |

**POSITION #6**




**NO REDOUBLE/TAKE**
**(JACOBY PARADOX)**

The analysis for this position is similar to that of Position #4. White has 26 rolls that will bear off his two checkers, and 10 rolls that miss (all rolls containing an ace other than double-aces). With White having 10 rolls that miss, Black will have greater than 25% winning chances if he gives White the cube and then misses, so he will have a take of White's redouble to eight. The trouble is that in this particular case it is very costly for Black to take that redouble, since he will lose so frequently – in fact, if White had only 9 rolls that missed (just one less), Black should pass the redouble.

As it turns out, when you run the numbers, Black shouldn't even turn the cube here. By doing so, he avoids getting hit with an eight-cube that he'll have to take in a position where he's a big underdog. Basically, if Black redoubles to four, all of White's wins will be on an eight-cube, while a large majority of Black's wins will be on a four-cube (although some will also be on an eight-cube). Since Black owns the cube on two, if he elects not to redouble, he ensures that all of the games will be decided on a two-cube; by doing this, he takes the best advantage of his approximate 2-to-1 game winning advantage. The equity calculations bear this out. Black will win $0.636 per game if he holds onto the cube, but if he redoubles, he'll only win $0.432 per game.

**EQUITIES – POSITION #6**

**IF BLACK DOUBLES**

| | |
|---|---|
| Black Winning Pct. | 65.90% |
| White Winning Pct. | 34.10% |
| Black $ per Game | $0.432 |

**IF BLACK HOLDS THE CUBE**

| | |
|---|---|
| Black Winning Pct. | 65.90% |
| White Winning Pct. | 34.10% |
| Black $ per Game | $0.636 |

## SUMMARY RECAP – WE DON'T WANT TO MISS THE FOREST FOR THE TREES

This problem set should have proven to be most interesting to those with backgrounds in finance, wealth management or mathematics, as well as to the gamblers in the audience. What most players will do in tackling this problem set is to start by recognizing that Black's winning chances get better as White's likelihood of bearing off his checkers declines. So far, so good. Next, these players will usually make their best guess as to where Black's "doubling point" is, and then assign a cube action of double/take (or double/pass) to those positions that are better for Black than the "doubling point" position, and assign a cube action of no double/take to those positions that are worse for Black. While this line of reasoning seems to be eminently reasonable and sound – *i.e.*, the greater Black's overall winning chances are, the greater the likelihood that doubling is the best play – and while it will usually get you to the best decision, it is in fact quite wrong in this particular case. And it is because of this mistaken line of thinking that many people are surprised when they are presented with the solutions for this problem set. Of course, the solutions, taken as a whole, are certainly counter-intuitive, to say the least, but suffice it to say that it is not sufficient to simply consider winning chances in these problems – that is to miss the forest for the trees. The guiding principle in making a doubling decision is not one of determining Black's winning chances and doubling accordingly; instead, the primary goal is to maximize the equity that Black can realize by either doubling or holding onto the cube, and to do that you simply must factor in the value of cube ownership as a component of that equity. Consideration must be given to how much Black sacrifices by forfeiting cube ownership.

Some have used the word "paradoxical" to describe a subset of these results. But the truth is that these problems all boil down to questions of risk and reward, and each is completely solvable with certainty through the application of basic mathematics, including some elementary probability theory. In each case, the player on roll is seeking to maximize his financial expectation. While he has no control over the fate that the dice Gods may have in store for him, the pertinent probabilities are known quantities, and the one thing that he does have some control over is the level of the stakes of the contest – since he owns the doubling cube. With cube ownership, the holder can offer to double the stakes at any time; the cost of doing so, though, is the forfeiture of cube ownership to the opponent. This has two main consequences. First, when you redouble, you forfeit the ability to utilize the cube at a later point in the game, and this could obviously end up costing you. Second, you give control over the cube to your opponent. This means that it will be your opponent that has all of the benefits associated with cube ownership – in particular, the exclusive right, exercisable at any later point in time, to compel you to either drop the game or to play for a further doubled level of stakes. A related benefit of cube ownership is that the owner of the cube will always have the ability to elect to play the game to conclusion. At times, this can be quite important.

This problem set essentially covers six different and distinct types of situations that can come about with these problem-types (for the categorical breakdown, see the equity chart that follows this summary recap). To begin, sometimes you will have cube ownership in what amounts to a pure "last roll" situation **(Position #1)**. In this case, your decision is simple – if you are the favorite, double and play for twice the stakes, and if you are the underdog, simply hold onto the cube to minimize your expected losses. Here, you simply have nothing to lose by doubling as a favorite. Other times, though, you will reach a favorable position where both sides have retained some winning chances that go beyond the initial roll. In these cases, you need to consider the advantages of doubling the stakes against the cost you incur by forfeiting your cube ownership to your opponent.

In some cases, by redoubling you might leave yourself in a situation where your opponent has a reasonable chance of obtaining a position where he will have a strong enough advantage that you will be forced to pass his redouble. When this happens, you'll be forced to surrender in a position where you have some residual winning chances. Sometimes the value you gain by having redoubled earlier on when you were a favorite will be more than sufficient to compensate you for those winning chances that you surrender when your opponent gets his opportunity to double you out; at other times, those forfeited winning chances will overshadow the gains you realize from having doubled the stakes earlier on as a favorite. Obviously, in the former cases you should exercise your right to redouble at the outset **(Position #2)**, and in the latter cases you should hold onto the cube, retaining the possibility of using it instead on your next turn **(Position #3)**. This may seem somewhat paradoxical, as you are holding onto the cube in some positions where your winning chances are greater than they are in other positions where you will be exercising your right to redouble. But that's precisely the point!! What you're trying to do is ensure that you'll be able to retain and realize upon those greater winning chances – and in these cases, the way to do that is to hold onto the cube so you don't find yourself being doubled out in a situation where you have decent winning chances. It's all about maximizing your overall financial equity so that you can win the most money.

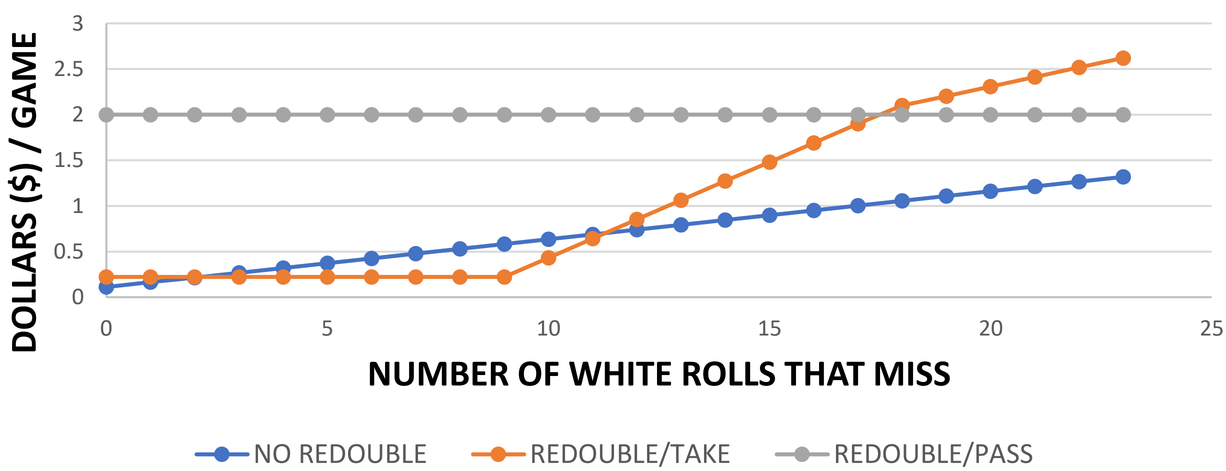
The possibility that you may have to forfeit winning chances is not the only potential cost that you can incur if you surrender the doubling cube to your opponent. At times, your opponent will turn the tables and become the favorite, but not such a large favorite that he can double you out and "cash" the game, causing you to forfeit some of your winning chances. Sometimes he will instead be able to "double you in" – meaning that he might be able to offer you a double that you should take, although he will only be doing this in a position where he himself is the favorite. That is, you will have a proper take, but it won't be a particularly happy one.

When facing such a possible scenario, you must once again evaluate the costs and benefits – risks and rewards. On the plus side, by redoubling as the favorite, you realize the benefit of increasing the stakes when you have the better chance of winning the game (this is especially obvious in the positions presented here, where most of your wins come on the very next shake). On the minus side, you leave your opponent with the option to elect to increase the stakes even further if he gains the upper hand, which will cost you in the form of larger potential financial losses in those games that are played for higher stakes. Sometimes, the benefits from redoubling will exceed the costs that you incur when your opponent finds an opportunity to redouble you and play for even higher stakes; in other cases, the opposite will be true, and your costs from redoubling will exceed the benefits. This is all largely dependent upon how big of a favorite your opponent becomes, but once again, your best course of action is to redouble at the outset when the prospective benefits outweigh the prospective costs **(Position #4)**, and to hold onto the cube at the outset when the prospective costs outweigh the prospective benefits **(Position #6)**. By managing the doubling cube in this manner, you'll be acting optimally to maximize your overall financial equity so that you can win the most money.

Finally, sometimes your winning chances will be large enough that your opponent won't even be able to profitably take a redouble **(Position #5)**. In these cases, unless there are other reasons to hold onto the cube – *e.g.*, cases where you might be able to win a gammon for a bigger payoff – you should just redouble and win the game on the spot. Giving your opponent a free chance to win a game that you should win immediately is obviously a fairly large mistake. On your better days, you'll be facing an opponent that is willing to mistakenly take your recube and you'll realize even greater financial rewards.

| Positions Ordered by Number of White Rolls That Miss | Number of White Rolls That Miss | Effect of a Possible White Redouble | Dollars ($) / Game if Black Redoubles | Dollars ($) / Game if Black Holds onto the Cube |
|---|---|---|---|---|
| Position 1 | 0 | n/a – **Last Roll Position** | $0.222 | $0.111 |
| Position 2 | 2 | Black is **Doubled OUT** | $0.222 | $0.216 |
| Position 3 | 7 | Black is **Doubled OUT** | $0.222 | $0.478 |
| Position 6 | 10 | Black is **Doubled IN** | $0.432 | $0.636 |
| Position 4 | 13 | Black is **Doubled IN** | $1.062 | $0.793 |
| Position 5 | 21 | White has a **Pass** | $2.000 | $1.213 |

## EQUITY GRAPH




## REFERENCES


Jacoby, Oswald, and Crawford, John R., ***The Backgammon Book***, 1970, The Viking Press, pp. 116-117.

Kleinman, Danny, ***Vision Laughs at Counting with Advice to the Dicelorn***, Lulu.com, 2007, pp. 176-182. (Note that the first edition of this book was published in 1980.)

Robertie, Bill, ***Advanced Backgammon***, The Gammon Press, 1991, Volume 2: Technical Play, Problem 358, pp. 210-211. (Note that the original version of this book was published as a single volume in 1984; it was subsequently expanded and re-released as a two-volume set in 1991.)